VIII. *An Account of a Book, intituled*, Harmonia Menſurarum, ſive Analyſis & Syntheſis per Rationum & Angulorum menſuras promotæ: accedunt alia Opuſcula Mathematica: per *Rogerum Coteſium*. Edidit & auxit *Robertus Smith*, Coll. Trin. Cantab. & Reg. Soc. Socius; Aſtronomiæ & Experimentalis Philoſoph. poſt *Coteſium* Profeſſor. *Cantabrigiæ* 1722. in 4to. Proſtant apud Bibliopolas Londinenſes.

THE Book conſiſts of three Parts. In the firſt, called *Logometria*, the Author's chief Deſign is to ſhew how that ſort of Problems, which are uſually reduced to the Quadrature of the *Hyperbola* and *Ellipſis*, may be reduced to the the Meaſures of Ratio's and Angles; and afterwards be ſolved more readily by the Canons of Logarithms and Sines and Tangents. He defines the Meaſures of Ratio's to be quantities of any kind, whoſe Magnitudes are analogous to the Magnitudes of the Ratio's to be meaſured. In this Senſe any Canon of Logarithms is a Syſtem of numeral Meaſures of the Ratio's of the abſolute Numbers to an Unit: The Parts of the Aſymptote of the Logiſtic Line, intercepted between its Ordinates, are a Syſtem of Linear Meaſures of the Ratio's of thoſe Ordinates: The Areas of an *Hyperbola*, intercepted between its Ordinates to the Aſymptote, are a Syſtem of Plane Meaſures of the Ratio's of thoſe Ordinates: And ſince there may be infinite Syſtems of Meaſures according as as various kinds of Quantities are made uſe of, ſuch as Numbers, Time, Velocity, and the like; or according as the Meaſures of any one Syſtem may be all increaſed or diminiſhed in any given proportion; in ſuch Variety much Confuſion may poſſibly ariſe as to the Kind and abſolute Magnitudes of particular Meaſures, which happen to fall under Conſideration. Our Author very happily removes this Difficulty; by ſhewing that the Nature of the Subject points out the Meaſure of a certain immutable Ratio for a *Modulus* in all Syſtems, whereby to determine the Kind and abſolute Magnitudes of all other Meaſures in each Syſtem.

The firſt Propoſition is to find the Meaſure of any propoſed Ratio. This he conſiders in a way ſo ſimple and general, as naturally leads to the Notion and Definition of a *Modulus*; namely, that it is an invariable Quantity in each Syſtem, which bears the ſame Proportion to the Increment of the Meaſure of any propoſed Ratio, as the increaſing Term of the Ratio bears to its own Increment. He then ſhews, that the Meaſure of any given Ratio is as the *Modulus* of the Syſtem, from whence it is taken: and that the *Modulus* in every Syſtem is always equal to the Meaſure of a certain determinate and immutable Ratio, which he therefore calls the *Ratio Modularis*. He ſhews that this Ratio is expreſſed by theſe Numbers 2,7182818 &c. to 1, or by 1 to 0,3678794 &c. So that in *Briggs*'s Canon the Logarithm of this Ratio is the *Modulus* of that Syſtem: In the Logiſtic Line the given Subtangent is the *Modulus* of that Syſtem: In the *Hyperbola* the given Parallelogram, contained by an Ordinate to the Aſymptote and the Abſciſs from the Center, is the *Modulus* of that Syſtem: and in other Syſtems the *Modulus* is generally ſome remarkable Quantity. In the ſecond Propoſition he gives a

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concise uncommon Method for calculating *Briggs's* Canon of Logarithms; together with Rules for finding intermediate Logarithms and Numbers, even beyond the Limits of the Canon. In the 3d Propofition he conftructs any Syftem of Meafures by a Canon of Logarithms; not only when the Meafure of fome one Ratio is given, but alfo without that *Datum*, by feeking the *Modulus* of the Syftem by the Rule abovementioned. In the 4th, 5th, and 6th Propofitions he fquares the *Hyperbola*, defcribes the Logiftic Line and Æquiangular Spiral by a Canon of Logarithms, and fhews fome curious Ufes of thefe Propofitions in their *Scholia*. Take an eafy Example of the Logometrical Method, in the common Problem for finding the Denfity of the Atmofphære. Suppofing Gravity uniform, every one knows, that if Altitudes are taken in any Arithmetical Progreffion, the Denfities of the Air in thofe Altitudes will be in a Geometrical Progreffion; that is, the Altitudes are the Meafures of the Ratio's of the Denfities below and in thofe Altitudes, and fo the difference of any two Altitudes is the Meafure of the Ratio of the Denfities in thofe Altitudes. Now to determine the abfolute or real Magnitude of thefe Meafures, the Author fhews, *a priori*, that the *Modulus* of the Syftem is the Altitude of the Atmofphære, when reduced every where to the fame Denfity as below. The *Modulus* therefore is given (as bearing the fame Proportion to the Altitude of the Mercury in the Barometer, as the fpecific Gravity of Mercury does to the fpecifick Gravity of Air) and confequently the whole Syftem is given. For fince in all Syftems the Meafures of the fame Ratio's are analogous among themfelves; the Logarithm of the Ratio of the Air's Denfity in any two Altitudes will be to the *Modulus* of the Canon, (that is, to the Logarithm of

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the *Ratio Modularis* defined above,) as the difference of thoſe Altitudes is to the aforeſaid given Altitude of the homogeneous Atmoſphære.

He concludes the Logometria with a *General Scholium*, containing great Variety of elegant Conſtructions both Logometrical and Trigonometrical; ſuch as give the Length of Curves either Geometrical or Mechanical; their Area's and Centers of Gravity; the Solids generated from them, and the Surfaces of theſe Solids; together with ſeveral curious Problems in Natural Philoſophy, concerning the Attraction of Bodies, the Denſity and Reſiſtance of Fluids, and the Trajectories of Planets. Several of theſe Problems have two Caſes; one conſtructed by the Meaſure of a Ratio, and the other by the Meaſure of an Angle. The great Affinity and beautiful *Harmony* of the *Meaſures* in theſe Caſes, has given occaſion to the Title of the Book. The Meaſures of Angles are defined (juſt as the Meaſures of Ratio's) to be Quantities of any Kind, whoſe Magnitudes are analogous to the Magnitudes of the Angles. Such may be the Arcs or Sectors of any Circle, or any other Quantities of Time, Velocity, or Reſiſtance, analogous to the Magnitudes of the Angles. Every Syſtem of theſe Meaſures has likewiſe its *Modulus* homogeneous to the Meaſures in that Syſtem, and may be computed by the Trigonometrical Canon of Sines and Tangents, juſt as the Meaſures of Ratio's by the Canon of Logarithms; for the given *Modulus* in each Syſtem bears the ſame Proportion to the Meaſure of any given Angle, as the *Radius* of a Circle bears to an Arc which ſubtends that Angle, or the ſame as this conſtant Number of Degrees 57,2957795130 bears to the Number of Degrees in the ſaid Angle. Upon the whole our Author thus expreſſes himſelf, *p.* 35. " Ex

" adductis

" adductis hactenus exemplis, Geometris integrum erit
" de methodo nostrâ judicare ; quam quidem, si pro-
" ba fuerit, ulterius excolere pergent & excolendo la-
" tius promovebunt. Patet utique campus amplissi-
" mus in quo vires suas experiri poterunt, præsertim si
" Logometriæ Trigonometriam insuper adjungant,
" quibus miram quandam affinitatem in se invicem
" euntibus intercedere notabam. Hisce quidem prin-
" cipiis haud facile crediderim *generaliora* dari posse ;
" cum tota Mathesis vix quicquam in universo suo am-
" bitu complectatur præter Angulorum & Rationum
" Theoriam. Neque sane *commodiora* sperabit, qui
" animadverterit effectionis facilitatem per amplissi-
" mas illas, omnibusque suis numeris absolutas, tum
" Logaritmmorum, tum Sinuum & Tangentium tabu-
" las ; quas antecessorum nostrorum laudatissimæ
" solertiæ debemus acceptas. Ut vero tanti beneficii
" uberior nobis exsurgat fructus, id nunc exponendum
" restat, quibus artibus ad istius modi conclusiones re-
" ctissimâ perveniatur. In hunc finem *Theoremata*
" quædam tum *Logometrica* tum *Trigonometrica* ad-
" jecissem, quæ parata ad usum asservo ; ni consul-
" tius visum esset, quum absque nimiis ambagibus ea
" tradi non possent, intacta potius præterire atque
" aliis denuo investiganda relinquere.

Why the Author takes his Principles to be so general, will farther appear by an Instance or two. In the Problem already mentioned he measures the Ratio of the Air's Densities in any Altitudes, by the Altitudes themselves, making use of the Altitude of an uniform Atmosphære for the *Modulus*. So likewise when he considers the Velocities acquired, and the Spaces described in given Times, by a Body projected upwards or downwards in a resisting Medium with any given Velocity ; he shews, that the Times of Descent, added to a given Time, are the Meaſures of Ratio's, to a given *Modulus* of Time, whoſe Terms are the Sum and Difference of the ultimate Velocity and the preſent Velocities that are acquired: that the Times of Aſcent, taken from a given Time, are the Meaſures of Angles, to a given *Modulus* of Time, whoſe Radius is to their Tangents in the Ratio of the ultimate Velocity to the preſent Velocities: and laſtly, that the Spaces deſcribed in Deſcent or Aſcent, are the Meaſures of Ratio's to a given *Modulus* of Space, whoſe Terms are the abſolute accelerating and retarding Forces ariſing from Gravity and Reſiſtance taken together at the Beginning and End of thoſe Spaces.

This general Account may ſuffice to illuſtrate what I am going to ſay; that ſince the Magnitudes of Ratio's (as well as their Terms) may be expounded by Quantities of any Kind, the Mathematician is at Liberty upon all Occaſions to chuſe thoſe which are fitteſt for his Purpoſe; and ſuch are they without doubt, that are put into his Hand by the Conditions of the Problem. He may indeed repreſent theſe Quantities by an *Hyperbola*, or any other Logometrical Syſtem, were not his Purpoſe anſwer'd with greater Simplicity by the very Syſtem itſelf, which occurs in each particular Problem. And the ſame may be ſaid for the Syſtems of Angular Meaſures, inſtead of recurring upon all Occaſions to Elliptical or Circular Area's.

As to the Convenience of calculating from our Author's Conſtructions, he ſhews that the Meaſures of any Ratio's or Angles are always computed in the ſame uniform Way; by taking from the Tables the Logarithm of the Ratio, or the Number of Degrees in the Angle, and then by finding a fourth Proportional to three given Quantities; for that will be the Meaſure

required

required. The ſimpleſt Hyperbolic Area may indeed be ſquared by the ſame Operation taught in the Author's fourth Propoſition; but the ſimpleſt Elliptic Area requires ſomewhat more: Thoſe that are more complex in both Kinds (which generally happens) require an additional Trouble to reduce them to the ſimpleſt: to ſquare them by infinite Series is ſtill more operoſe, and does not anſwer the End of Geometry. Upon the whole therefore it may deſerve to be conſidered, for what Purpoſes ſhould Problems be always conſtructed by Conic Areas, unleſs it be to pleaſe or aſſiſt the Imagination. The Deſign of Theoretical Geometry differs from Problematical; the former conſiſts in the Diſcovery and Contemplation of the Properties and Relations of Figures for the ſake of naked Truth; but the Deſign of the latter is to do ſomething propoſed, and is beſt executed by the leaſt *Apparatus* of the former.

The *Logometria* was firſt publiſhed by the Author himſelf, in the *Philoſoph. Tranſact.* of the Year 1714. No 338. But his Logometrical and Trigonometrical Theorems abovementioned were not publiſhed till after his Deceaſe. Theſe Theorems make the ſecond Part of the Book, and are calculated to give the Fluents of Fluxions (reduced to 18 Forms) by Meaſures of Ratio's and Angles; in ſuch a manner, that any Perſon may perfectly comprehend their Conſtruction and Uſe, though altogether unacquainted with Curvilinear Figures, as expreſſed by Æquations. And this Circumſtance does alſo render the Application of them to the *Analyſis* and *Conſtruction* of *Problems* extremely eaſy. Of this kind the Author has given a great many choice Examples both in abſtract and phyſical Problems; which make up the third and laſt Part of the Book.

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The Author, a little before his Deceaſe has informed us (in a Letter of *May* 5. wrote to his Friend Mr. *Jones*) " that Geometers had not yet promoted the inverſe " Method of Fluxions, by Conic Areas, or by Mea- " ſures of Ratio's and Angles, ſo far as it is capable of " being promoted by thoſe Methods. There is an " infinite Field (ſays he) ſtill reſerved, which it has " been my Fortune to find an Entrance into. Not to " keep you longer in Suſpenſe, I have found out a ge- " neral and beautiful Method by Meaſures of Ratio's " and Angles for the Fluent of any Quantity which " can come under this Form $\frac{d\dot{z}z^{\theta n+\frac{\delta}{\lambda}n-1}}{e+fz^n}$, in " which d, e, f are any conſtant Quantities, z the vari- " able, n any Index, θ any whole Number affirmative or " negative, $\frac{\delta}{\lambda}$ any Fraction whatever. The Fluents of " this Form which have hitherto been conſidered are $\frac{d\dot{z}z^{\theta n-1}}{e+fz^n}$ & $\frac{d\dot{z}z^{\theta n+\frac{1}{2}n-1}}{e+fz^n}$: Theſe you remem- " ber are Sir *Iſaac Newton*'s two firſt, and from " theſe all his others are eaſily deduced. And as his " irrational Forms of the quadratick Kind are derived " from the rational, ſo from my general rational " Form I deduce irrational ones of all Kinds. For in- " ſtance, if $\frac{\delta}{\lambda}$ repreſent any affirmative or negative " Fraction, the Fluent of any Quantity of this Form " $d\dot{z}z^{\theta n-1} \times \overline{e+fz^n}|^{\frac{\delta}{\lambda}}$, or of this $d\dot{z}z^{\theta n-1}$ " $\times \overline{\frac{e+fz^n}{g+hz^n}}|^{\frac{\delta}{\lambda}}$ and ſo of ſome others, depends upon the

" the Meaſures of Ratio's and Angles. Mr. *Leibnitz* " in the *Leipſic Acts* of 1702, *p.* 218 and 219, has ve- " ry raſhly undertaken to demonſtrate, that the Fluent " of $\frac{\dot{x}}{x^4 + a^4}$ cannot be expreſſed by Meaſures of Ra- " tio's and Angles; and he ſwaggers upon the Occa- " ſion (according to his uſual Vanity) as having by " this Demonſtration determined a Queſtion of the " greateſt Moment. Then he goes on thus; as the " Fluent of $\frac{\dot{x}}{x + a}$ depends upon the Meaſure of a " Ratio, and the Fluent of $\frac{\dot{x}}{xx + aa}$ upon the Meaſure " of an Angle; ſo he had more than once expreſſed " his Wiſhes, that the Progreſſion may be continued, " and it be determined to what Problem the Fluents of " $\frac{\dot{x}}{x^4 + a^4}$, $\frac{\dot{x}}{x^8 + a^8}$, &c. may be referred. His De- " ſire is anſwered in my general Solution, which " contains an infinite Number of ſuch Progreſſions. " I can go yet farther, and ſhew him how by Mea- " ſures of Ratio's and Angles, without any Exception " or Limitation, the Fluent of this general Quantity " $\frac{d\dot{z}z^{\theta n + \frac{\delta}{\lambda} n - 1}}{e + fz^n + gz^{2n}}$ or even this $\frac{d\dot{z}z^{\theta n + \frac{\delta}{\lambda} n - 1}}{e + fz^n + gz^{2n} + hz^{3n}}$ " may be had; where θ, as before, repreſents any Inte- " ger, and the Denominator λ of the Fraction $\frac{\delta}{\lambda}$, re- " preſents any Number in this Series, 2. 4. 8. 16. 32. &c. " any whole Number being denoted by its Numerator " δ. In truth I am inclined to believe, that Mr. *Leib-* " *nitz*'s grand Queſtion ought to be determined

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" the contrary Way; and that it will be found at laſt,
" that the Fluent of any rational Fluxion whatever,
" does depend upon the Meaſures of Ratio's and
" Angles, excepting thoſe which may be had in finite
" Terms even without introducing Meaſures.

Dr. *Taylor* knowing by this Letter what the Author had done, was pleaſed to propoſe the Invention of the Fluents of the two laſt Fluxions as a Problem to the Mathematicians in foreign Parts. Mr. *Bernoulli* in the *Leipſic Acts* of 1719. *p.* 256, did ſhew accordingly how they are reducible to Conic Area's. The Editor has publiſhed the Author's own Solution by Meaſures of Ratio's and Angles; and upon this Foundation has conſtructed new Tables of Logometrical and Trigonometrical Theorems, for the Fluents of Fluxions reduced to 94 Forms, part rational and part irrational. He has likewiſe added general Notes upon the chief Difficulties in the Book, together with a Method of compoſing Synthetical Demonſtrations of Logometrical and Trigonometrical Conſtructions, illuſtrated by various Examples.

The firſt Treatiſe in the *Miſcellaneous Works* is concerning the *Eſtimation of Errors in Mixt Mathematicks.* It conſiſts of 28 Theorems, to determine the Proportions among the leaſt contemporary Variations of the Sides and Angles of Plane and Sphærical Triangles, while any two of them remain invariable. An Example will ſhew their great Uſe in Aſtronomy. The Time of the Day or Night is frequently to be determined by the Altitude of ſome Star. Let it then be propoſed to find the Error, that may ariſe in the Time, from any given Error in taking the Altitude. By applying the 22d Theorem to the Triangle form'd by the Complements of the Star's Altitude and Declination

and

and by the Complement of the Pole's Elevation, the Author ſhews, that the Variation of the Angle at the Pole, and conſequently the Error in Time, will be as the Error in the Altitude directly, as the Sine Complement of the Pole's Elevation inverſely, and as the Sine of the Star's *Azimuth* from the Meridian inverſely. Conſequently, if the Error in the Altitude be given, under a given Elevation of the Pole, the Error in Time will be reciprocally as the Sine of the *Azimuth* contained by the Meridian and the Vertical which the Star is in. This Error therefore will be the ſame, whatever be the Altitude of the Star in the ſame Vertical; and will be leaſt when the Vertical is at right Angles to the Meridian. But will be abſolutely the leaſt in the ſame Circumſtance, if the Obſerver be under the Æquator. In which Caſe, if the Error in the Altitude be one Minute, the Error in the Time will be four Seconds, If the Obſerver recedes from the Æquator towards either Pole, the Error will be increaſed in the Proportion of the Radius to the Sine Complement of the Latitude: So that in the Latitude of 45 Degrees it will be $5\frac{2}{3}$ Seconds, and in the Latitudes of 50 and 55 it will be $6\frac{2}{9}$ and $6\frac{37}{38}$ Seconds reſpectively. If the Star be in any other vertical Oblique to the Meridian, the Error will ſtill be increaſed in the Proportion of the Radius to the Sine of that oblique Angle. Laſtly, if the Error in the Altitude be either bigger or leſs than one Minute, the Error in Time will be bigger or leſs in the ſame Proportion. Much after the ſame manner may the Limits of Errors be computed in other Caſes, which ariſe from the Inaccuracy of Obſervations, and from hence the moſt convenient Opportunities for obſerving are alſo determined.

The Second Treatiſe is concerning the *Differential Method.* The Author having wrote it, before he had feen Sir *Isaac Newton's* Treatife upon that Subject, has handled it after a manner fomewhat different.

The Title of the Third Treatife is *Canonotechnia* or concerning the Conftruction of Tables by Differences. It confifts of ten Propofitions, moft admirably contrived for expeditious Computation of intermediate Terms in any given Series. The laft Propofition, which contains a general Solution of the whole Defign, is this; *Datis feriei cujufcunque terminis aliquot æquidiftantibus, quorum intervalla fecanda funt in æquales quotlibetcunq; partes, propofitum fit invenire terminos interferendos.*

The Book concludes with three fmall Tracts, concerning the Defcent of Bodies, the Motion of Pendulums in the Cycloid, and the Motion of Projectiles, compofed in a very natural and eafy manner.

The Author has wrote fome other Pieces, yet unpublifh'd, which the Editor has given an Account of in his Preface to the Book.

The Reader will find every Subject treated with uncommon Elegance and Simplicity.

FINIS.

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